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Static interactions of non-abelian vortices

Roberto Auzzi
Department of Physics, Swansea University,
Singleton Park, Swansea SA2 8PP, U.K.
E-mail: r.auzzi@swan.ac.uk

Minoru Eto and Walter Vinci
INFN, Sezione di Pisa,
Largo Pontecorvo, 3, Ed. C, 56127 Pisa, Italy, and
Department of Physics, University of Pisa,
Largo Pontecorvo, 3, Ed. C, 56127 Pisa, Italy
E-mail: minoru@df.unipi.it, walter.vinci@pi.infn.it

Abstract: Interactions between non-BPS non-Abelian vortices are studied in non-Abelian $\mathrm{U}(1) \times \mathrm{SU}(N)$ extensions of the Abelian-Higgs model in four dimensions. The distinctive feature of a non-Abelian vortex is the presence of an internal $\mathbf{C}P^{N-1}$ space of orientational degrees of freedom. For fine-tuned values of the couplings, the vortices are BPS and there is no net force between two static parallel vortices at arbitrary distance. On the other hand, for generic values of the couplings the interactions between two vortices depend non-trivially on their relative internal orientations. We discuss the problem both with a numerical approach (valid for small deviations from the BPS limit) and in a semi-analytical way (valid at large vortex separations). The interactions can be classified with respect to their asymptotic property at large vortex separation. In a simpler fine-tuned model, we find two regimes which are quite similar to the usual type I/II Abelian superconductors. In the generic model we find other two new regimes: type I*/II*. Unlike the type I (type II) case, where the interaction is always attractive (repulsive), the type I* and II* have both attractive and repulsive interactions depending on the relative orientation. We have found a rich variety of interactions at small vortex separations. For some values of the couplings, a bound state of two static vortices at a non-zero distance exists.

Keywords: Solitons Monopoles and Instantons, Nonperturbative Effects.

Contents

1. Introduction

Nobody doubts the importance of topological solitons in various areas of modern physics (see [1] for a general review). They are closely related to the phenomena of spontaneous symmetry breaking which e.g. occur as a phase transition from the high temperature phase of the early universe to the present cold universe. In particular, vortex strings are believed to play important roles in the confinement of quarks in QCD, and they could be relevant to the study of cosmic string effects in the early universe. Historically, the vortex string, as a topological soliton, was found in the Abelian-Higgs model by Abrikosov and Nielsen-Olesen [2, 3].

Recently, a new type of vortex was found in $\mathrm{U}(N)$ non-Abelian gauge theories coupled with $N_f = N$ Higgs fields in the fundamental representation [4, 5]. This object is called non-Abelian vortex. A typical feature of this vortex is that it possesses internal degrees of freedom, which arise when the vortex breaks an exact flavor symmetry of the vacuum. These degrees of freedom are related to the orientations of the non-Abelian flux inside the vortex core. The vacuum of the theory leaves the colour-flavour locked $\mathrm{SU}(N)_{\mathrm{C+F}}$ symmetry; on the other hand, the vortex soliton spontaneously breaks this $\mathrm{SU}(N)_{\mathrm{C+F}}$ symmetry to $\mathrm{SU}(N-1)_{\mathrm{C+F}} \times \mathrm{U}(1)_{\mathrm{C+F}}$; these broken symmetries give rise to the moduli space of an elementary vortex

$$\mathbf{C}P^{N-1} = \frac{\mathrm{SU}(N)_{\mathrm{C+F}}}{\mathrm{SU}(N-1)_{\mathrm{C+F}} \times \mathrm{U}(1)_{\mathrm{C+F}}}.$$

The classical moduli coordinate can be promoted to a field living in the vortex worldvolume; in this way vortex solitons in a $3+1$ dimensional theory are directly connected with a $\mathbf{C}P^{N-1}$ sigma model in $1+1$ dimension, which describes the macroscopic physics of the flux tube. Several groups have intensively investigated these objects in relations to various aspects of physics; a partial list includes confined monopoles [6], quantum aspects [7–9], higher winding numbers [10–12], relation to D-branes in string theory [4, 13], dualities [14–16], cosmic strings [17, 18], semilocal extensions [19, 20], $\mathrm{SO}(N)$ generalization [21], high temperature QCD [22], global vortices [23], gravity [24], composite states of various BPS solitons [6, 25] and statistical mechanics [26]. Readers can find good reviews in [27, 28].

Most the works on the non-Abelian vortex so far were focused on the BPS limit [29] (a single non-Abelian non-BPS vortex configuration is discussed in [30, 31, 14, 22]). No forces arise among BPS vortices, because there is a nice balance between the repulsive forces mediated by the vector particles and attractive forces mediated by the scalar particles. In this particular limit, the solutions to the equations of motion develop a full moduli space of solutions [32]. However, once the balance between the attractive force and the repulsive force is lost, the moduli space disappears. Alternatively we can think that an effective potential is generated on this moduli space. It is well known that ANO vortices in the type I system feel an attractive force while those in the type II model feel a repulsive force [29, 33–36, 38]. In condensed matter physics, it is also known that type II vortices form the so-called Abrikosov lattice, [2, 39] due to the repulsive force between them. Furthermore, lattice simulations give some evidence of the presence of a (marginal) type II superconductivity in QCD [40].

We are interested in studying interactions between non-Abelian vortices which are non-BPS. In non-supersymmetric theories, BPS configurations are obtained with fine-tuned values of the couplings. If supersymmetry exists in the real world, it is surely broken at a low energy scale; therefore non-BPS vortices are more natural than BPS ones. Also, we encounter such non-BPS configurations in supersymmetric theories [5, 12, 16] when we consider a hierarchical symmetry breaking closely related to a dual picture of color confinement of truly non-Abelian kind. Specifically we are interested in the interactions between vortices with different internal orientations, which is the distinct feature from the ANO case. In a previous paper [41], we have discussed these aspects in an $\mathcal{N} = 2$ theory with an adjoint mass term which breaks the extended supersymmetry, and we have found a natural non-Abelian generalization of type I superconductors. Even if the force between two non-Abelian vortices is not always attractive, we have found a close resemblance with type I Abelian vortices: the lightest field of the theory is a scalar field. So if we put two vortices at large distance, the prevailing part of the interaction is mediated by the scalar particles and not by vector particles. Moreover, if the two vortices have the same orientation in the internal moduli space, the force is always attractive.

In this paper we study the same problem in another theoretical setting: an extension of the Abelian-Higgs model with arbitrary scalar couplings which is generically incompatible with the BPS limit. The simplest extension in this direction is a theory in which there are just two mass scales; the mass of the vector bosons and the mass of the scalars. There is one parameter λ which controls the ratio of the two mass scales. We find that $\lambda < 1$ leads to an attractive force as a usual Abelian type I, while for $\lambda > 1$ a repulsive force works, similarly to the usual Abelian type II. There is no force between vortices with opposite $\mathbf{C}P^{N-1}$ orientation. For $\lambda < 1$ (type I) this configuration is unstable and the true minimum of the potential corresponds to two coincident vortices with the same orientation. For $\lambda > 1$ (type II) this configuration is stable; in other words a part of the moduli space corresponding to the relative distance between vortices with opposite orientations survives the non-BPS perturbation.

However, in more general theories where the masses of the Abelian and non-Abelian degrees of freedom are different, we find a more complicated picture. There are four mass scales, the masses of the $\mathrm{U}(1)$ and of the $\mathrm{SU}(N)$ vector bosons, the masses of the scalars in the adjoint representation and the singlet of $\mathrm{SU}(N)_{\mathrm{C+F}}$. At large distance, the interaction between two vortices is dominated by the particle with the lightest mass. So if we keep the four masses as generic parameters, at large vortex separation we find four different regimes that we call Type I, Type II, Type I* and Type II*. In the last two categories repulsive and attractive interactions depend on the relative orientation. We study also numerically the interactions among two vortices at any separation with arbitrary orientations, and find that short distance forces also have rich qualitative features depending both on the relative orientations and the relative distance.

The paper is organized as follows. In section 2 we describe the theoretical set-up. In section 3 we write the equations for the vortex and we quickly review the moduli space of the two vortices in the BPS limit. In section 4 vortices in a fine-tuned setup are studied; the effective vortex potential in the case of small deviations from the BPS limit is found numerically. In section 5 a more general set-up with four independent parameters is discussed in the same way. In section 6 the effective potential at large vortex separation is found using a semi-analytical approach. Section 7 contains the conclusions. In the appendix we provide the link between the formalism of this paper and that of the companion paper [41].

2. Theoretical set-up

2.1 A fine-tuned model

Our natural starting point is the following non-Abelian, $\mathrm{U}(N)$, extension of the Abelian-

JHEP02(2008)100

Higgs model in four dimensions:

$$\mathcal{L} = \mathrm{Tr}\left[-\frac{1}{2g^2}F_{\mu\nu}F^{\mu\nu} + \mathcal{D}_\mu H(\mathcal{D}^\mu H)^\dagger - \frac{\lambda^2 g^2}{4}\left(v^2\mathbf{1}_N - HH^\dagger\right)^2\right]. \tag{2.1}$$

Here, for simplicity we take the same gauge coupling g for both the $\mathrm{U}(1)$ and $\mathrm{SU}(N)$ groups, while $\lambda^2 g^2/4$ is a scalar coupling and v (>0) determines the Higgs VEV. In this simple model we have only three couplings (g, λ, v). The N by N matrix field H embodies N Higgs fields in the fundamental representation of $\mathrm{U}(N)$. There is also an $\mathrm{SU}(N)$ flavor symmetry which acts on H from the right hand side. The vacuum of the model is given by:

$$HH^\dagger = v^2\mathbf{1}_N. \tag{2.2}$$

The vacuum breaks completely the $\mathrm{U}(N)$ gauge symmetry, although a global color-flavor locking symmetry $\mathrm{SU}(N)_{\mathrm{C+F}}$ is preserved

$$H \to U_{\mathrm{G}} H U_{\mathrm{F}}, \quad U_{\mathrm{G}} = U_{\mathrm{F}}^\dagger, \quad U_{\mathrm{G}} \in \mathrm{SU}(N)_{\mathrm{G}},\ U_{\mathrm{F}} \in \mathrm{SU}(N)_{\mathrm{F}}. \tag{2.3}$$

The trace part $\mathrm{Tr}H$ is a singlet under the color-flavor group and the traceless parts are in the adjoint representation. We have two mass scales, one for the vector bosons and the other for the scalar bosons. The $\mathrm{U}(1)$ and the $\mathrm{SU}(N)$ gauge vector bosons have both the same mass

$$M_{\mathrm{U}(1)} = M_{\mathrm{SU}(N)} = g\,v. \tag{2.4}$$

The masses of the scalars are given by the eigenvalues of the mass matrix. We start with $2N^2$ real scalar fields in H: N^2 of them are eaten by the gauge bosons (the Higgs mechanism) and the other N^2 (one singlet and the rest adjoint) have same masses

$$M_{\mathrm{s}} = M_{\mathrm{ad}} = \lambda\, g\, v. \tag{2.5}$$

When we choose the critical coupling $\lambda = 1$ (BPS), the mass of these scalars is the same as the mass of the gauge bosons and the Lagrangian allows an $\mathcal{N} = 2$ supersymmetric extension. The BPS vortices saturating the BPS energy bound admit infinitely degenerate set of solutions.

2.2 Models with general couplings

A straightforward generalization of the fine-tuned model (2.1) is to consider different gauge couplings, e for the $\mathrm{U}(1)$ part and g for the $\mathrm{SU}(N)$ part, and a slightly more general scalar potential

$$\mathcal{L} = \mathrm{Tr}\left[-\frac{1}{2g^2}\hat{F}_{\mu\nu}\hat{F}^{\mu\nu} - \frac{1}{2e^2}f_{\mu\nu}f^{\mu\nu} + \mathcal{D}_\mu H(\mathcal{D}^\mu H)^\dagger\right] - V, \tag{2.6}$$

where we have defined $\hat{F}_{\mu\nu} = \sum_{A=1}^{N^2-1} F_{\mu\nu}^A T_A$ and $f_{\mu\nu} = F_{\mu\nu}^0 T^0$ with $\mathrm{Tr}(T^A T^B) = \delta^{AB}/2$ and $T^0 = \mathbf{1}/\sqrt{2N}$ [1]. The scalar potential is:

$$
\begin{aligned}
V &= \frac{\lambda_g^2 g^2}{2} \sum_{A=1}^{N^2-1} \left(H^{i\dagger} T^A H_i \right)^2 + \frac{\lambda_e^2 e^2}{4N} \left(H^{i\dagger} H_i - N v^2 \right)^2 \\
&= \frac{\lambda_g^2 g^2}{4} \mathrm{Tr} \hat{X}^2 + \frac{\lambda_e^2 e^2}{4} \mathrm{Tr} \left(X^0 T^0 - v^2 \mathbf{1}_N \right)^2 ,
\end{aligned} \tag{2.7}
$$

where

$$
X \equiv H H^\dagger = X^0 T^0 + \sum_{A=1}^{N^2-1} X^A T^A, \quad \hat{X} \equiv \sum_{A=1}^{N^2-1} X^A T^A = 2 \sum_{A=1}^{N^2-1} \left(H^{i\dagger} T^A H_i \right) T^A. \tag{2.8}
$$

The Lagrangian has the same symmetries as the previous fine-tuned model (2.1). The potential in eq. (2.7) is the most general gauge invariant quartic potential which can be built with the matter content of the theory. The U(1) and the SU(N) vector bosons have different masses

$$
M_{\mathrm{U}(1)} = e\, v, \qquad M_{\mathrm{SU}(N)} = g\, v. \tag{2.9}
$$

Moreover, the singlet part of H has a mass M_{s} different from that of the adjoint part M_{ad}

$$
M_{\mathrm{s}} = \lambda_e\, e\, v, \qquad M_{\mathrm{ad}} = \lambda_g\, g\, v. \tag{2.10}
$$

When we take equal couplings, $g = e$ and $\lambda \equiv \lambda_e = \lambda_g$, the scalar potential reduces to the simple potential $V_{g=e} = \frac{\lambda^2 g^2}{4} \mathrm{Tr} \left(X - v^2 \mathbf{1}_N \right)^2$. For the critical values $\lambda_e = \lambda_g = 1$, the Lagrangian allows an $\mathcal{N} = 2$ supersymmetric extension and then the model admits BPS vortices which saturate the BPS energy bound.

3. Non-Abelian vortices in the fine-tuned model

3.1 Vortex equations

We study the fine-tuned model (2.1) through out this section. For convenience, let us make the following rescaling of fields and coordinates:

$$
H \to v H, \quad W_\mu \to g v W_\mu, \quad x_\mu \to \frac{x_\mu}{g v}. \tag{3.1}
$$

The Lagrangian then in eq. (2.1) takes the form

$$
\tilde{\mathcal{L}} = \frac{\mathcal{L}}{g^2 v^4} = \mathrm{Tr} \left[-\frac{1}{2} F_{\mu\nu} F^{\mu\nu} + \mathcal{D}_\mu H (\mathcal{D}^\mu H)^\dagger - \frac{\lambda^2}{4} \left(\mathbf{1}_N - H H^\dagger \right)^2 \right], \tag{3.2}
$$

and the masses of vector and scalar bosons are rescaled to

$$
M_{\mathrm{U}(1)} = M_{\mathrm{SU}(N)} = 1, \qquad M_{\mathrm{s}} = M_{\mathrm{ad}} = \lambda. \tag{3.3}
$$

[1] In the case $g = e$ it is more compact to use $F_{\mu\nu} = \hat{F}_{\mu\nu} + f_{\mu\nu}$.

As explained in the introduction, the model with $\lambda < 1$ ($\lambda > 1$) in the Abelian case ($N = 1$) is called type I (type II) and the forces between vortices are attractive (repulsive). At the critical coupling $\lambda = 1$, there are no forces between vortices, so that multiple vortices stably coexist.

In order to construct non-BPS non-Abelian vortex solutions, we have to solve the following 2nd order differential equations, derived from the Lagrangian (2.1),

$$\mathcal{D}_\mu F^{\mu\nu} - \frac{i}{2}\left[H(\mathcal{D}^\nu H)^\dagger - (\mathcal{D}^\nu H)H^\dagger\right] = 0, \tag{3.4}$$

$$\mathcal{D}_\mu \mathcal{D}^\mu H + \frac{\lambda^2}{4}\left(1 - HH^\dagger\right) H = 0. \tag{3.5}$$

From now on, we restrict ourselves to static configurations depending only on the coordinates x^1, x^2. Here we introduce a complex notation

$$z = x^1 + ix^2, \quad \partial = \frac{\partial_1 - i\partial_2}{2}, \quad W = \frac{W_1 - iW_2}{2}, \quad \mathcal{D} = \frac{\mathcal{D}_1 - i\mathcal{D}_2}{2} = \partial + iW. \tag{3.6}$$

The equation of motions are of course not gauge invariant but covariant. It might be better to study gauge invariant quantities instead of dealing with the original fields H and W_μ. For this purpose we rewrite our fields as follows

$$\bar{W}(z,\bar{z}) = -iS^{-1}(z,\bar{z})\bar{\partial}S(z,\bar{z}), \quad H(z,\bar{z}) = S^{-1}(z,\bar{z})\tilde{H}(z,\bar{z}), \tag{3.7}$$

where S takes values in $GL(N,\mathbf{C})$ and it is in the fundamental representation of $\mathrm{U}(N)$ while the gauge singlet $\tilde{H}$ is an $N \times N$ complex matrix. There is an equivalence relation $(S,\tilde{H}) \sim (V(z)S, V(z)\tilde{H})$, where $V(z)$ is a holomorphic $GL(N,\mathbf{C})$ matrix with respect to z, because different elements in the same equivalence class give us the same physical fields as in eq. (3.7). The gauge group $\mathrm{U}(N)$ and the flavor symmetry act as follows

$$S(z,\bar{z}) \to U_{\mathrm{G}} S(z,\bar{z}), \quad H_0(z) \to H_0(z)U_{\mathrm{F}}, \qquad U_{\mathrm{G}} \in \mathrm{U}(N)_{\mathrm{G}},\ U_{\mathrm{F}} \in \mathrm{SU}(N)_{\mathrm{F}}. \tag{3.8}$$

In order to write down the equations of motion (3.4) and (3.5) in a gauge invariant fashion, we introduce a gauge invariant quantity

$$\Omega(z,\bar{z}) \equiv S(z,\bar{z})S(z,\bar{z})^\dagger. \tag{3.9}$$

With respect to the gauge invariant objects Ω and $\tilde{H}$, the equations (3.4) and (3.5) are written in the following form

$$4\bar{\partial}^2\left(\Omega\partial\Omega^{-1}\right) - \tilde{H}\bar{\partial}\left(\tilde{H}^\dagger\Omega^{-1}\right) + \bar{\partial}\tilde{H}\tilde{H}^\dagger\Omega^{-1} = 0, \tag{3.10}$$

$$\Omega\partial\left(\Omega^{-1}\bar{\partial}\tilde{H}\right) + \bar{\partial}\left(\Omega\partial\left(\Omega^{-1}\tilde{H}\right)\right) + \frac{\lambda^2}{4}\left(\Omega - \tilde{H}\tilde{H}^\dagger\right)\Omega^{-1}\tilde{H} = 0. \tag{3.11}$$

Notice that eq. (3.10) is a 3rd order differential equation. This is the price we have to pay in order to write down the equations of motion in terms of gauge invariant quantities. These equations must be solved with the following boundary conditions for k vortices:

$$\det\tilde{H} \to z^k, \quad \Omega \to \tilde{H}\tilde{H}^\dagger, \qquad \text{as} \quad z \to \infty. \tag{3.12}$$

The field strength is given by

$$F_{12} = 2S^{-1}\bar{\partial}\left(\Omega\partial\Omega^{-1}\right)S. \tag{3.13}$$

Notice that eq. (3.10) is invariant under the SU(N) flavor symmetry while eq. (3.11) is covariant. This leads to Nambu-Goldstone zero modes for vortex solutions.

3.2 BPS limit

To see the relation with the BPS equations, let us take a holomorphic function $\tilde{H}$ with respect to z as

$$\tilde{H} = H_0(z). \tag{3.14}$$

Then the equations (3.10) and (3.11) reduce to

$$\bar{\partial}\left[4\bar{\partial}\left(\Omega\partial\Omega^{-1}\right) - H_0H_0^\dagger\Omega^{-1}\right] = 0, \tag{3.15}$$

$$\bar{\partial}\left(\Omega\partial\Omega^{-1}\right) + \frac{\lambda^2}{4}\left(\mathbf{1} - H_0H_0^\dagger\Omega^{-1}\right) = 0. \tag{3.16}$$

These two equations are consistent only in the BPS limit $\lambda = 1$. The equation (3.16) is the master equation for the BPS non-Abelian vortex and the holomorphic matrix $H_0(z)$ is called the moduli matrix [10, 28]. For any given moduli matrix $H_0(z)$, given the corresponding solution to the master equation, the physical fields W_μ and H are obtained via eq. (3.7). All the complex parameters contained in the moduli matrix are moduli of the BPS vortices. For example, the position of the vortices can be read from the moduli matrix as zeros of its determinant $\det H_0(z_i) = 0$. Furthermore, the number of vortices (the units of magnetic flux of the configuration) corresponds to the degree of $\det H_0(z)$ as a polynomial with respect to z. The classification of the moduli matrix for the BPS vortices is given in ref. [10, 28].

From now on, we consider a U(2) gauge theory ($N = 2$), which is the minimal model for non-Abelian vortices. The minimal winding BPS vortex is described by two moduli matrices

$$H_0^{(1,0)} = \begin{pmatrix} z - z_0 & 0 \\ -b' & 1 \end{pmatrix}, \qquad H_0^{(0,1)} = \begin{pmatrix} 1 & -b \\ 0 & z - z_0 \end{pmatrix}. \tag{3.17}$$

The complex parameter z_0 corresponds to the position of the vortex while the other parameters, b and b', parameterize the internal orientation. This modulus gives rise to an internal moduli space $\mathbf{C}P^1$ [10, 28]. In fact, they are inhomogeneous coordinates for $\mathbf{C}P^1$ and are related by the transition function $b = 1/b'$. This orientational modulus can easily be understood from a simple argument related to the symmetry of the theory as we will see below.

A rigorous way to define the orientation of the non-Abelian vortex is to identify it with the null eigenvector of $H_0(z)$ at the vortex position $z = z_0$. For the moduli matrix in eq. (3.17) the orientational vectors are

$$\vec{\phi}^{\,(1,0)} = \begin{pmatrix} 1 \\ b' \end{pmatrix} \quad \sim \quad \vec{\phi}^{\,(0,1)} = \begin{pmatrix} b \\ 1 \end{pmatrix}. \tag{3.18}$$

Here "$\sim$" stands for an identification up to complex non zero factors: $\vec{\phi} \sim \lambda\vec{\phi}$, $\lambda \in \mathbf{C}^*$. One can easily find a direct relation between the parameter b (b') and the broken $\mathrm{SU}(2)_{\mathrm{C+F}}$. Since H_0 transforms as $H_0 \to H_0 U_{\mathrm{F}}$ under the color-flavor group, the orientational vector $\vec{\phi}$ transforms as $\vec{\phi} \to U_{\mathrm{F}}^\dagger \vec{\phi}$. If we start with $\vec{\phi} = (1,0)^T$, we can recover $\vec{\phi} = (1,b')^T$ by use of the color-flavor rotation as

$$\vec{\phi} = U_{\mathrm{F}}^\dagger \vec{\phi} \quad \Leftrightarrow \quad \begin{pmatrix} 1 \\ 0 \end{pmatrix} \to \begin{pmatrix} \alpha^* & -\beta \\ \beta^* & \alpha \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \sim \begin{pmatrix} 1 \\ \beta^*/\alpha^* \end{pmatrix} \tag{3.19}$$

with $|\alpha|^2 + |\beta|^2 = 1$. Thus we identify b' and β^*/α^*. In what follows, we will call two non-Abelian vortices with equal orientational vectors *parallel*, while when they have orthogonal orientational vectors we will call them *anti-parallel*. The reader must keep in mind that vortices are always parallel in real space. Throughout this paper, we use the words parallel and anti-parallel only referring to the internal orientational vectors.

Generic configurations of two vortices at arbitrary positions and with arbitrary orientations are described by the moduli matrices [10, 28]:

$$H_0^{(1,1)} = \begin{pmatrix} z - \phi & -\eta \\ -\tilde{\eta} & z - \tilde{\phi} \end{pmatrix}, \quad H_0^{(2,0)} = \begin{pmatrix} z^2 - \alpha' z - \beta' & 0 \\ -a' z - b' & 1 \end{pmatrix}. \tag{3.20}$$

The superscripts label patches covering the moduli space. One more patch similar to $(2,0)$ is needed to complete the full moduli space [10, 28]. The positions of the vortices are the roots of $z_i^2 - (\phi + \tilde{\phi}) z_i + \phi\tilde{\phi} - \eta\tilde{\eta} = z_i^2 - \alpha' z_i - \beta' = 0$. By using translational symmetry we can set $z_1 + z_2 = 0$ ($\phi + \tilde{\phi} = \alpha' = 0$) without loss of generality. The orientation vectors are $\vec{\phi}_1^{(1,1)} = (\eta,\ z_1 - \phi)^T$ and $\vec{\phi}_2^{(1,1)} = (\eta,\ z_2 - \phi)^T$ for the $(1,1)$ patch, while they are $\vec{\phi}_1^{(2,0)} = (1,\ a' z_1 + b')^T$ and $\vec{\phi}_2^{(2,0)} = (1,\ a' z_2 + b')^T$ for the $(2,0)$ patch. Overall complex factor does not have physical meaning, so that each vector takes value on $\mathbf{C}P^1$. We can describe anti-parallel vortices only in the (1,1) patch when $\eta = \tilde{\eta}$, because of $\vec{\phi}_1^\dagger \vec{\phi}_2 = 0$. On the other hand, we can describe parallel vortices only in the $(2,0)$ patch when $a' z_1 + b' = a' z_2 + b'$.

For convenience, let us take a special subspace where $\tilde{\eta} = 0$ in the $(1,1)$ patch:

$$H_{0\ \mathrm{red}}^{(1,1)} \equiv \begin{pmatrix} z - z_0 & -\eta \\ 0 & z + z_0 \end{pmatrix}, \quad z_0 = z_1 = -z_2 = \phi = -\tilde{\phi}. \tag{3.21}$$

One can always recover generic points in eq. (3.20) using flavor rotations. The parameters (z_0, η) and (β, a', b') are related by the following relations: $\beta' = z_0^2$, $a' = 1/\eta$ and $b' = -z_0/\eta$. The orientational vectors are then of the form

$$\vec{\phi}_1^{(1,1)}\big|_{z=z_0} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \vec{\phi}_2^{(1,1)}\big|_{z=-z_0} = \begin{pmatrix} \eta \\ -2z_0 \end{pmatrix}. \tag{3.22}$$

When the vortices are coincident, however, the rank of the moduli matrix at the vortex position reduces. In this case we can no longer define two independent orientations for each vortex but we can only define an overall orientation. In fact, one can see that when $z_0 = 0$, the two orientations in eq. (3.22) are both equal to $\vec{\phi}^{\,(1,1)} = (1,\ 0)^T$. We cannot really give to the parameter η an exact physical meaning of a relative orientation between two coincident vortices. It is better, in this case, to consider this parameter merely as an internal degree of freedom of the composite vortex. When we take correctly into account both the parameter η and the global flavour rotations that we previously factorized out, we recover the full moduli space for coincident vortices, which is $W\mathbf{C}P^2_{(2,1,1)} \simeq \mathbf{C}P^2/Z_2$ [17, 11, 12].

The definitions of the position and orientation of a single vortex can be rigorously extended to the non-BPS case by merely replacing $H_0(z)$ with $\tilde{H}_0(z,\bar{z})$. For configurations with several vortices, all the flat directions that are not related to Goldstone modes or translational symmetries will disappear. It is possible to use these definitions as constraints on the $\tilde{H}_0(z,\bar{z})$ matrix to fix positions and orientations. Then our formalism allows us to study the static interactions of non-BPS configurations.

4. Vortex interaction in the fine-tuned model

We now concentrate on the fine-tuned model (3.2). We will first calculate the masses of a special class of non-BPS coincident vortices. Then we will derive an effective potential for coincident almost BPS vortices but with generic value of the internal modulus parameter. Finally, we will compute an effective potential for two almost BPS vortices at any distance and with any relative orientations.

4.1 (k_1, k_2) coincident vortices

The minimal winding solution in the non-Abelian gauge theory is a mere embedding of the ANO solution into the non-Abelian theory. This is obvious also from the moduli matrix view point. In fact, in the non-Abelian moduli matrix (3.17) we can put b (or b') to zero with a global flavour rotation. Henceforth we can recognize the moduli matrix for the single ANO vortex: $H_0^{\mathrm{ANO}}(z) = z - z_0$ as the only non-trivial element of the moduli matrix.

This kind of embedding is also useful to investigate a simple non-BPS configuration. Let us start with the moduli matrix for a configuration of k coincident vortices. Since we have an axial symmetry around the k coincident vortices, we can make the following reasonable ansatz for Ω and $\tilde{H}$

$$\Omega^{(0,1)} = \begin{pmatrix} 1 & 0 \\ 0 & w(r) \end{pmatrix}, \qquad \tilde{H}^{(0,1)} = \begin{pmatrix} 1 & 0 \\ 0 & f(r) z^k \end{pmatrix}. \tag{4.1}$$

Note that $f(r) = 1$ means $\tilde{H}^{(0,1)} = H_0^{(0,1)}(z)$ which is nothing but the BPS solution. We will call the multiple vortex which is generated by the ansatz in eq. (4.1) "$(0,k)$-vortex". In terms of the two fields $w(r) = e^{Y(r)}$ and $f(r)$ the equations (3.10) and (3.11) reduce to the following form

$$Y'' + \frac{1}{r}Y' - \frac{2}{f}\left(f'' + \frac{1+2k}{r}f' - Y'f'\right) - \lambda^2\left(1 - r^{2k} f^2 e^{-Y}\right) = 0, \tag{4.2}$$

$$Y''' + \frac{1}{r}Y'' - \frac{1}{r^2}Y' + e^{-Y} r^{2k-1} f^2 \left(2k - rY'\right) = 0. \tag{4.3}$$

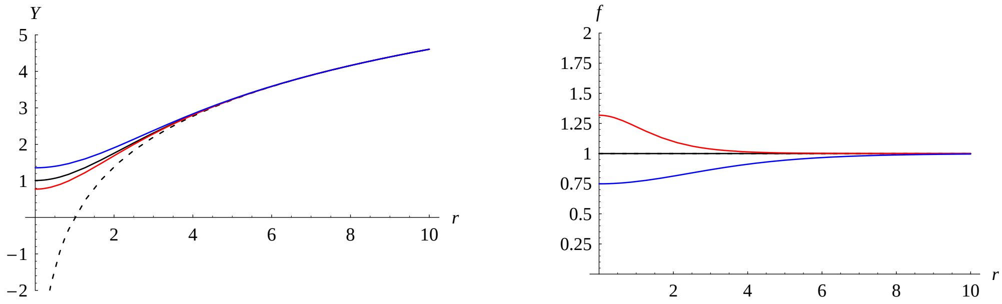


Figure 1: Numerical plots of Y (left) and f (right) for the single vortex: black for $\lambda = 1$, red for $\lambda = 1.7$, blue for $\lambda = 0.5$. The broken line is $2 \log r$.

λ	$k = 1$	$k = 2$	$k = 3$
0.6	0.81305	1.52625	2.21205
0.7	0.86440	1.65337	2.42101
0.8	0.91231	1.77407	2.62115
0.9	0.95737	1.88936	2.81382
1	1.00000	2.00000	3.00000
1.1	1.04053	2.10655	3.18045
1.2	1.07922	2.20944	3.35575
1.3	1.11626	2.30905	3.52639
1.4	1.15182	2.40566	3.69276

Table 1: Numerical value for the masses of coincident vortices.

The boundary conditions are

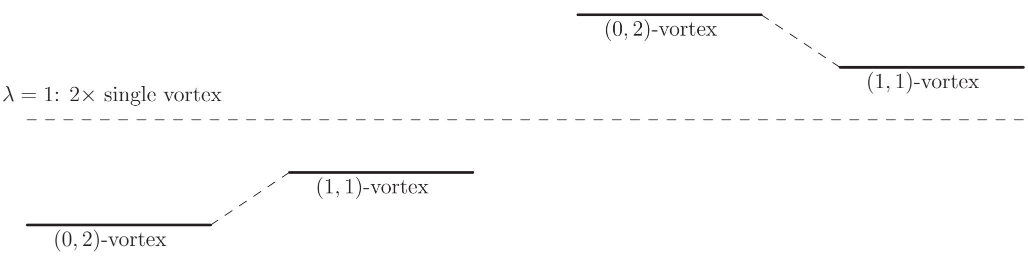
$$Y \to 2k \log r, \quad Y' \to 2k/r, \quad f \to 0 \qquad (r \to \infty); \tag{4.4}$$

$$Y' \to 0, \quad f' \to 0 \qquad (r \to 0). \tag{4.5}$$

Although it is quite impossible to solve these differential equation analytically, we can solve them numerically. The results for the single vortex ($k = 1$) are shown in figure 1, where we used several different values of λ .

When $k \geq 2$ it is possible that the ansatz (4.1) does not give the true solution (minimum of the energy) of the equations of motion (3.10) and (3.11). This is because there could be repulsive forces between the vortices. With ansatz (4.1) we fix the positions of all the vortices at the origin by hand. The reduced equations (4.2) and (4.3) are nevertheless still useful to investigate the interactions between two vortices. The results are listed in table 1.

For $\lambda = 1$, the masses are identical to integer values, up to 10^{-5} order, which are nothing but the winding number of the vortices. Furthermore, our numerical results for generic λ are in perfect agreement with the numerical value for ANO vortices obtained about 30 years ago by Jacobs and Rebbi [35]. As mentioned, this happens because the $(0, k)$-vortex is obtained by embedding of the k ANO vortices.



(a) type I ($\lambda < 1$) (b) type II ($\lambda > 1$)

Figure 2: Spectrum of the $(0, 2)$ and $(1, 1)$ coincident vortices.

There is another type of composite configuration which can easily be analyzed numerically. These configurations are generated by the following ansatz for Ω and $\tilde{H}$

$$\Omega^{(1,1)} = \begin{pmatrix} w_1(r) & 0 \\ 0 & w_2(r) \end{pmatrix}, \qquad \tilde{H}^{(1,1)} = \begin{pmatrix} f_1(r) z^{k_1} & 0 \\ 0 & f_2(r) z^{k_2} \end{pmatrix}. \tag{4.6}$$

This ansatz corresponds to a configuration with k_1 composite vortices which wind in the first diagonal $\mathrm{U}(1)$ subgroup of $\mathrm{U}(2)$ and with k_2 coincident vortices that wind the second diagonal $\mathrm{U}(1)$ subgroup. The two sets of vortices can be considered each as embedded ANO vortices for the two decoupled Abelian subgroups. We refer to these decoupled non-Abelian vortices as a "(k_1, k_2)-vortex". The mass of a (k_1, k_2)-vortex is thus the sum of the mass of the $(k_1, 0)$-vortex and that of the $(0, k_2)$-vortex. For example, the mass of $(1, 1)$-vortex is double of the mass of the $(0, 1)$-vortex listed in the first column of table 1. As in the previous case, we get the minima of the energy under the constraint that the vortices are coincident.

Because our fine-tuned non-Abelian model is a simple extension of the Abelian-Higgs model, we expect similar behavior for the interactions. Actually we have only one parameter λ. Thus we will call the non-Abelian vortices for $\lambda < 1$ type I, while they will be called type II for $\lambda > 1$. From figure 2, in which is summarized the relevant data of table 1, we can argue which kind of interaction appears between two non-Abelian vortices. In the type I case, the $(0, 2)$-vortex is energetically preferred to the $(1, 1)$-vortex, while in type II case the $(1, 1)$-vortex is preferred. If the two vortices are separated sufficiently, we can ignore any interaction between them. Regardless of their orientations, the mass of two well separated vortices is twice that of the single vortex. This mass is equal to the mass of the $(1, 1)$-vortex. Furthermore, it seems that the two separated anti-parallel vortices do not interact, and the energy does not depend on the relative distance. For the type I case, figure 2 suggests that the configuration with $(2, 0)$-vortices is the true minimum of the system. This means that attractive forces appears between vortices with different orientations. An attractive force also works in the internal space, which aligns the orientations. In the type II case, it seems that we do not have an isolated minimum of the energy. In fact, all the anti-parallel configurations with arbitrary distance have the same value of the energy. In the following sections we will confirm the picture we have outlined here.

4.2 Effective potential for coincident vortices

The dynamics of BPS solitons can be investigated by the so-called moduli approximation [32]. The effective action is a massless non-linear sigma model whose target space is the moduli space. The sigma model is obtained by plugging a BPS solution into the original Lagrangian and promoting the moduli parameters to massless fields, then picking up quadratic terms in the derivatives with respect to the vortex world-volume coordinates

$$L = \int dx^1 dx^2 \ \mathcal{L}\left[H_{\rm sol}(\varphi_i(t,x^3)), W^\mu_{\rm sol}(\varphi_i(t,x^3))\right]_{\lambda=1}, \tag{4.7}$$

where φ_i represents the set of moduli parameters $(\eta, \tilde{\eta}, \phi, \tilde{\phi})$ or $(\alpha', \beta', a', b')$ contained in the moduli matrix (3.20).

If the coupling constant λ is close to the BPS limit $\lambda = 1$, we can still use the moduli approximation, to investigate dynamics of the non-BPS non-Abelian vortices by adding a potential of order $|1-\lambda^2| \ll 1$ to the massless sigma model

$$L = \int dx^1 dx^2 \ \mathcal{L}\left[H_{\rm sol}(\varphi_i(t,x^3)), W^\mu_{\rm sol}(\varphi_i(t,x^3))\right]_{\lambda=1} - V(\varphi_i). \tag{4.8}$$

We shall now calculate the effective potential $V(\varphi_i)$ using the method suggested by Hindmarsh, who calculated this effective potential for non-BPS semilocal vortex in the Abelian-Higgs model [42].

First we write the Lagrangian (3.2) in the following way

$$\tilde{\mathcal{L}} = \tilde{\mathcal{L}}_{\rm BPS} + \frac{(\lambda^2-1)}{4}\left(\mathbf{1}_N - HH^\dagger\right)^2. \tag{4.9}$$

We get non-BPS corrections of order $O(\lambda^2-1)$ by putting BPS solutions into eq. (4.9). This is because the first term is minimized by the BPS solution, while the second one is already a term of order $O(\lambda^2-1)$. The energy functional thus takes the following form

$$\mathcal{E} = \frac{E}{2\pi v^2} = 2 + \frac{(\lambda^2-1)}{8\pi}\int dx^1 dx^2 \ {\rm Tr}\left(\mathbf{1} - H_{\rm BPS}(\varphi_i)H^\dagger_{\rm BPS}(\varphi_i)\right)^2 \tag{4.10}$$

where $H_{\rm BPS}(\varphi_i)$ stands for the BPS solution generated by the moduli matrices in eq. (3.20). The first term corresponds to the mass of two BPS vortices and the second term is the deviation from the BPS solutions which is nothing but the effective potential we want.

In this section we consider the effective potential on the moduli space of coincident vortices. To this end, it suffices to consider only the following matrices

$$H_0^{(1,1)} = \begin{pmatrix} z & -\eta \\ 0 & z \end{pmatrix}, \quad H_0^{(2,0)} = \begin{pmatrix} z^2 & 0 \\ -a'z & 1 \end{pmatrix}. \tag{4.11}$$

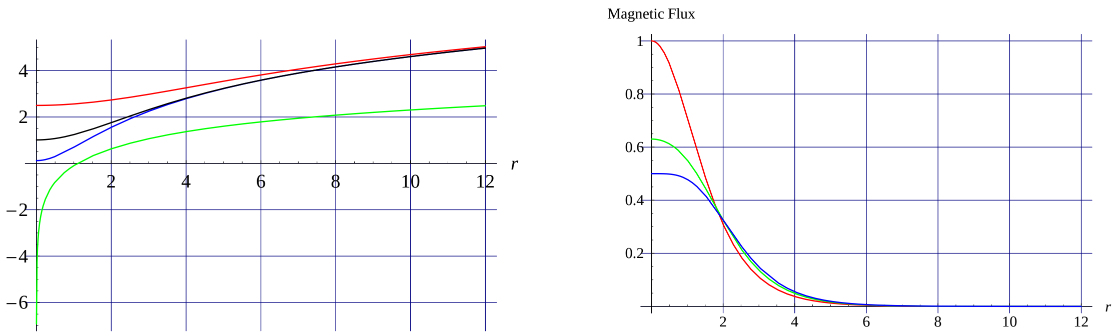


Figure 3: Left: Numerical plots of Y_1 (red), Y_2 (green) and Y_3 (blue) for $\eta = 3$. The black line is $Y_1 = Y_3$ for $\eta = 0$. Right: Magnetic flux $\mathrm{Tr}F_{12}$ with $\eta = 0$ (red), $\eta = 3$ (green) and $\eta = \infty$ ($a' = 0$) (blue).

The parameters η and a' are related by $\eta = 1/a'$. The effective potential on the moduli space for two coincident vortices is thus [2]

$$\frac{V(|\eta|)}{2\pi v^2} \equiv \frac{(\lambda^2-1)}{8\pi}\int dx^1 dx^2\ \mathrm{Tr}\left(\mathbf{1} - H_{\mathrm{BPS}}(|\eta|)H^\dagger_{\mathrm{BPS}}(|\eta|)\right)^2 \equiv (\lambda^2-1)\mathcal{V}(|\eta|), \quad (4.12)$$

where we have defined a reduced effective potential $\mathcal{V}$ which is independent of λ. To evaluate this effective potential, we need to solve the BPS equations for a composite state of two non-Abelian vortices with an intermediate value of η. Such numerical solutions found in [11]. We propose here another reliable technique for the numerics, which needs much less algebraic efforts.

In the moduli matrix formalism, what we should solve is only the master equation (3.16) for Ω with $\lambda = 1$ and $\tilde{H}(z,\bar{z}) = H_0(z)$. Because of the axial symmetry of the composite vortex and the boundary condition at infinity:

$$\Omega \to H_0(z)H_0^\dagger(\bar{z}), \quad (4.13)$$

we can make a simple ansatz for Ω. For example in the patch $(1,1)$ we can write

$$\Omega^{(1,1)} = \begin{pmatrix} w_1(r) & -\eta e^{-i\theta}w_2(r) \\ -\eta e^{i\theta}w_2(r) & w_3(r) \end{pmatrix}. \quad (4.14)$$

The advantage of the moduli matrix formalism is that only three functions $w_i(r)$ are needed and the formalism itself is gauge invariant. Plugging the ansatz into eq. (3.16), after some algebra we get the following differential equations

$$Y_1'' + \frac{1}{r}Y_1' + \frac{|\eta|^2}{r^2}\frac{(1+rY_1'-rY_2')^2}{|\eta|^2 - e^{Y_1+Y_3-2Y_2}} = 1 - e^{-Y_1}(r^2+|\eta|^2); \quad (4.15)$$

$$Y_2'' + \frac{1}{r}Y_2' - \frac{1}{r^2}\frac{(1+rY_1'-rY_2')(1+rY_2'-rY_3')}{1-|\eta|^2 e^{-Y_1-Y_3+2Y_2}} = 1 - e^{-Y_2}r; \quad (4.16)$$

$$Y_3'' + \frac{1}{r}Y_3' + \frac{|\eta|^2}{r^2}\frac{(1+rY_2'-rY_3')^2}{|\eta|^2 - e^{Y_1+Y_3-2Y_2}} = 1 - e^{-Y_3}r^2, \quad (4.17)$$

[2] The potential depends only on $|\eta|$ because the phase of η can be absorbed by the flavor symmetry. The same holds for the coordinate a'

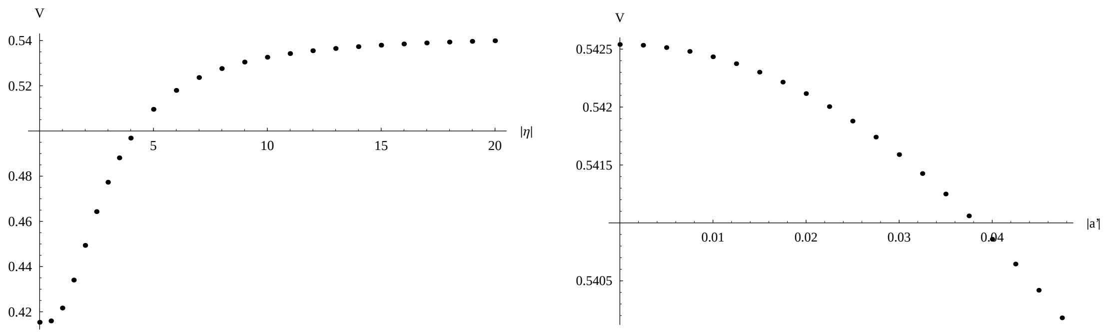


Figure 4: Numerical plots of the effective reduced potential $\mathcal{V}(|\eta|)$. In the second plot $a' = 1/\eta$.

λ	$(2,0)_{\rm num}$	$(2,0)_{\rm eff}$	$(1,1)_{\rm num}$	$(1,1)_{\rm eff}$
0.6	1.52625	1.65280	1.62611	1.73440
0.7	1.65337	1.72332	1.72880	1.78835
0.9	1.88936	1.89692	1.91473	1.92115
0.95	1.94523	1.94711	1.95793	1.95954
1	2.00000	2.00000	2.00000	2.00000
1.05	2.05376	2.05561	2.04102	2.04254
1.1	2.10655	2.11393	2.08106	2.08715
1.15	2.15843	2.17496	2.12018	2.13384
1.2	2.20944	2.23870	2.15843	2.18260

Table 2: Numerical value for the masses of coincident vortices. $(2,0)_{\rm num}$ is for the numerical results while $(2,0)_{\rm eff}$ is for our approximation using the effective potential.

where we have redefined the fields as $w_i(r) = e^{Y_i(r)}$ with $i = 1,2,3$. We solve numerically these differential equations using a simple relaxation method, see figure 3.

The effective potential can be obtained by plugging numerical solutions into eq. (4.12). The result is shown in figure 4. When we consider configurations with big values of $|\eta|$, it is better to switch to the other patch and use the variable a'. We can still make a similar ansatz for Ω, leading to simple differential equations like those in eqs. (4.15)$\sim$(4.17). In the left of figure 4 we show a numerical plot of the reduced effective potential $\mathcal{V}$ from $|\eta| = 0$ ($|a'| = \infty$) to $|\eta| = 20$. In the right of figure 4 we show another plot from $|a'| = 0$ ($|\eta| = \infty$) to $|a'| = 1/20$. To give an estimate of the range of validity of our approximation we can compare the results obtained in this section with the numerical integrations obtained for $(2,0)$-vortices and also $(1,1)$-vortices. The comparison is made in table 2 from which we can argue that the effective potential approximation gives result with an accuracy around 10% for the range of values $0.7 < \lambda < 1.15$.

In the type II case ($(\lambda^2 - 1) > 0$) the effective potential has the same qualitative behavior as showed in the figure. As we expected, it has a minimum at $|\eta| = 0$. This matches the previous result that the $(1,1)$-vortex is energetically preferred to the $(2,0)$-vortex. In the type I case ($(\lambda^2 - 1) < 0$) the shape of the effective potential can be

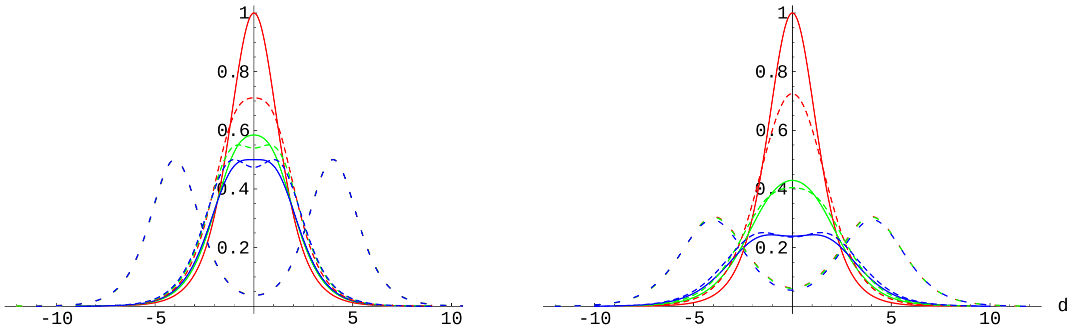


Figure 5: The magnetic flux $\mathrm{Tr}F_{12}$ of a configuration of 2 BPS vortices. The figure shows slices including the centers of the vortex. (red, green, blue) correspond to $\eta = (0, 4, \infty)$ while (solid, small broken, wide broken) lines to $d = (0, 1, 4)$. The left panel shows the fine-tuned model with $e = g$ and the right shows the model with $e = 2g$.

obtained just by flipping the overall sign of the effective potential of the type II case. Then the effective potential always takes a negative value, which is consistent with the fact that the masses of the type I vortices are less than that of the BPS vortices, see the middle column in table 1. Contrary to the type II case, the type I potential has a minimum at $|a'| = 0$ ($|\eta| = \infty$). This means that the $(2, 0)$-vortex is preferred with respect to the $(1, 1)$ vortex.

4.3 Interaction at generic vortex separation

In this subsection we go on investigating the interactions of non-Abelian vortices in the U(2) gauge group at generic distances. As in section 4.2 we will use the moduli space approximation, considering only small deviations from the BPS case. The generic configurations are described by the moduli matrices in eq. (3.20). We will consider here only the reduced $(1, 1)$ patch defined in eq. (3.21). By putting the two vortices on the real axis we can reduce z_0 to a real parameter d. Furthermore, by the flavor symmetry, we can freely put $\tilde{\eta} = 0$ and suppress the phase of η. The relevant configurations will be described by the following moduli matrix:

$$H_{0\ \mathrm{red}}^{(1,1)} = \begin{pmatrix} z - d & -\eta \\ 0 & z + d \end{pmatrix}, \tag{4.18}$$

where $2d$ is the relative distance and η the relative orientation.

Now let us study the effective potential as function of η and d. We first need the numerical solution to the BPS master equation for two vortices with any relative distance and orientation. Unlike the computation for the coincident vortices, we do not have an axial symmetry. We can no longer reduce the problem to one spatial dimension by making an appropriate ansatz. Nevertheless the moduli matrix formalism is a powerful tool also for the numerical calculations. The master equation is a 2 by 2 hermitian matrix, so it includes four real 2nd order partial differential equations. Despite the great complexity of this system of coupled equations, the relaxation method is very effective to solve the

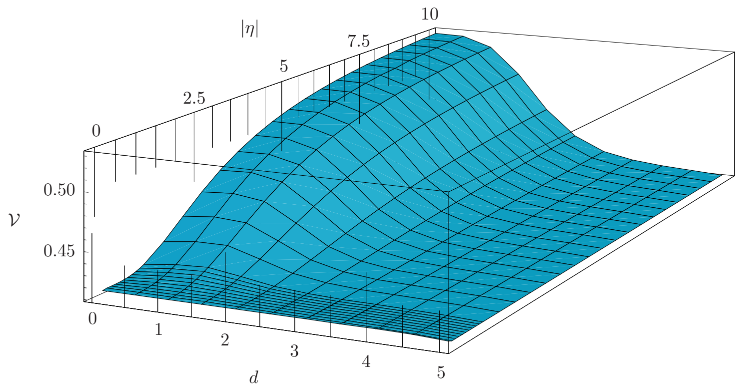


Figure 6: Numerical plot of the reduced effective potential $\mathcal{V}(\eta, d)$.

problem. We show several numerical solutions in figure 5. As before, once we get the numerical solution to the BPS equations, the effective potential is obtained by plugging them into eq. (4.10).

The numerical plot of the reduced effective potential $\mathcal{V}$ is shown in figure 6. The effective potential for the type II case has the same shape, up to a small positive factor $(\lambda^2 - 1)$. The potential forms a hill whose top is at $(d, |\eta|) = (0, \infty)$. It clearly shows that two vortices feel repulsive forces, in both the real and internal space, for every distance and relative orientation. The minima of the potential has a flat direction along the d-axis where the orientations are anti-parallel ($\eta = 0$) and along the η axis at infinite distance ($d = \infty$). Therefore the anti-parallel vortices do not interact.

In the type I case ($\lambda < 1$) the effective potential is upside-down of that of the type II case. There is unique minimum of the potential at $(d, |\eta|) = (0, \infty)$. This means that attractive force works not only for the distance in real space but also among the internal orientations. Configurations with anti-parallel orientations do not interact, but these configurations represent unstable points of equilibrium. Type I vortices always stick together.

5. Vortices with generic couplings

In this section we will shift from the fine-tuned $\mathrm{U}(N)$ model (2.1) to the more general model defined in eqs. (2.6) and (2.7). This will lead to more complicated algebra, but we will also clarify interactions with different qualitative behaviors. There are 5 parameters $(g, e, \lambda_g, \lambda_e, v)$ in the model, but we can reduce their number with the following rescaling

$$H \to vH, \quad W_\mu \to evW_\mu, \quad x_\mu \to \frac{x_\mu}{ev}. \tag{5.1}$$

Then the Lagrangian (2.6) is expressed as follows

$$\tilde{\mathcal{L}} = \mathrm{Tr}\left[-\frac{1}{2\gamma^2}\hat{F}_{\mu\nu}\hat{F}^{\mu\nu} - \frac{1}{2}f_{\mu\nu}f^{\mu\nu} + \mathcal{D}_\mu H(\mathcal{D}^\mu H)^\dagger\right] - \frac{\gamma^2\lambda_g^2}{4}\mathrm{Tr}\hat{X}^2 - \frac{\lambda_e^2}{4}\mathrm{Tr}\left(X^0T^0 - \mathbf{1}_N\right)^2, \tag{5.2}$$

where $\tilde{\mathcal{L}} = \mathcal{L}/e^2 v^4$ and we have introduced the ratio of the two gauge couplings

$$\gamma \equiv \frac{g}{e}. \tag{5.3}$$

We have three effective couplings $\gamma, \lambda_e, \lambda_g$ and the rescaled masses of particles are

$$M_{\mathrm{U}(1)} = 1, \quad M_{\mathrm{SU}(N)} = \gamma, \quad M_{\mathrm{s}} = \lambda_e, \quad M_{\mathrm{ad}} = \gamma \lambda_g. \tag{5.4}$$

The Lagrangian can be thought of as the bosonic part of a supersymmetric model only when both the parameters λ_e and λ_g are unity. In the BPS case we can easily find the BPS equations

$$\bar{\mathcal{D}} H = 0, \qquad \hat{F}_{12} = \frac{\gamma^2}{2}\hat{X}, \qquad F_{12}^0 T^0 = \frac{1}{2}\left(X^0 T^0 - \mathbf{1}_N\right). \tag{5.5}$$

The last two equations can be rewritten in a compact way

$$F_{12} = \frac{1}{2}\left(X - \mathbf{1}_N\right) + \frac{\gamma^2 - 1}{2}\hat{X}. \tag{5.6}$$

The first BPS equation in eq. (5.5) can be solved using the moduli matrix in the usual way

$$H = S^{-1}(z, \bar{z}) H_0(z), \qquad \bar{W} = -i S^{-1} \bar{\partial} S, \tag{5.7}$$

where S is a $GL(N, \mathbf{C})$ matrix. We would like to stress that this solution does not depend on γ so that the moduli space of the BPS vortices is the same as that of the well investigated vortices in the equal gauge coupling theory $g = e$.[3] The eq. (5.6) can be rewritten in a gauge invariant fashion as

$$\bar{\partial}\left(\Omega \partial \Omega^{-1}\right) = \frac{1}{4}\left(\Omega_0 \Omega^{-1} - \mathbf{1}_N\right) + \frac{\gamma^2 - 1}{4}\left(\Omega_0 \Omega^{-1} - \frac{\mathrm{Tr}\left(\Omega_0 \Omega^{-1}\right)}{N}\mathbf{1}_N\right) \tag{5.8}$$

where $\Omega = S S^\dagger$ is same as before and $\Omega_0 \equiv H_0 H_0^\dagger$.

Now we are ready to investigate interactions between two almost BPS vortices by using same strategy as we used in section 4. An effective action of the moduli dynamics for appropriately small $|1 - \lambda_{e,g}^2| \ll 1$ is obtained by plugging BPS solutions into the action. Then we get

$$\begin{aligned} \frac{V(\eta, d; \gamma, \lambda_e, \lambda_g)}{2\pi v^2} &= \int d\tilde{x}^2 \left(\frac{\gamma^2(\lambda_g^2 - 1)}{8\pi}\mathrm{Tr}\hat{X}^2 + \frac{\lambda_e^2 - 1}{8\pi}\mathrm{Tr}\left(X^0 T^0 - \mathbf{1}_N\right)^2\right) \\ &= \frac{1}{2\pi}\int d\tilde{x}^2\ \mathrm{Tr}\left[\frac{\lambda_g^2 - 1}{\gamma^2}(\hat{F}_{12})^2 + (\lambda_e^2 - 1)(F_{12}^0 T^0)^2\right], \end{aligned} \tag{5.9}$$

where we have used the BPS equations in the second line. Let us define the Abelian and the non-Abelian potentials as

$$\mathcal{V}_e(\eta, d; \gamma) = \int d\tilde{x}^2\ \mathrm{Tr}(F_{12}^0 T^0)^2, \qquad \mathcal{V}_g(\eta, d; \gamma) = \int d\tilde{x}^2\ \mathrm{Tr}(\hat{F}_{12})^2. \tag{5.10}$$

The true potential is a linear combination of them

$$V(\eta, d; \gamma, \lambda_e, \lambda_g) = (\lambda_e^2 - 1)\mathcal{V}_e(\eta, d; \gamma) + \frac{\lambda_g^2 - 1}{\gamma^2}\mathcal{V}_g(\eta, d; \gamma). \tag{5.11}$$

Notice that $\mathcal{V}_{e,g}$ is determined by the BPS solutions, so it does not depend on $\lambda_{e,g}$.

[3] The moduli space is the same from the topological point of view, while the metric will be different.

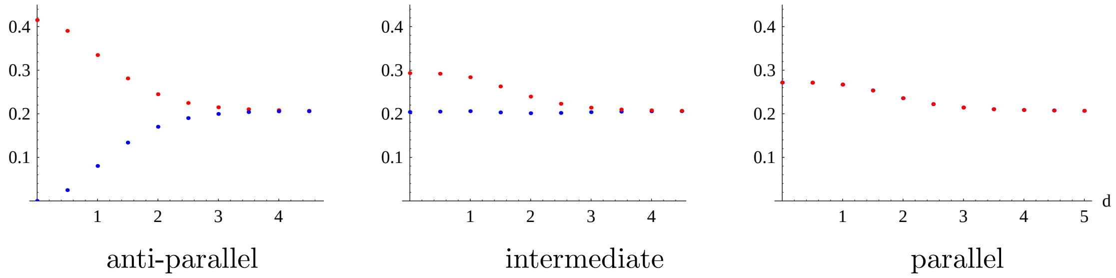


Figure 7: Effective potential vs. distance of vortices d. Red dots denote $\mathcal{V}_e$ while blue denote $\mathcal{V}_g$. All plots are for $\gamma = 1$. The left figure shows the two anti-parallel vortices $\eta = 0$, the middle shows $\eta = 4.1$ and the right shows the two parallel vortices $\eta = \infty$.

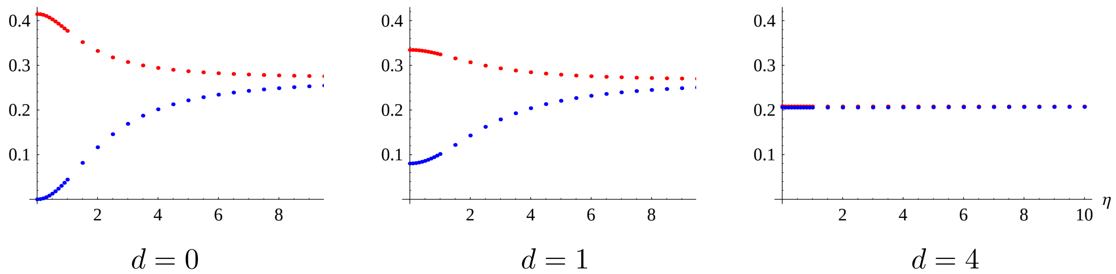


Figure 8: Effective potential vs. η. Red dots denote $\mathcal{V}_e$ while blue denote $\mathcal{V}_g$.

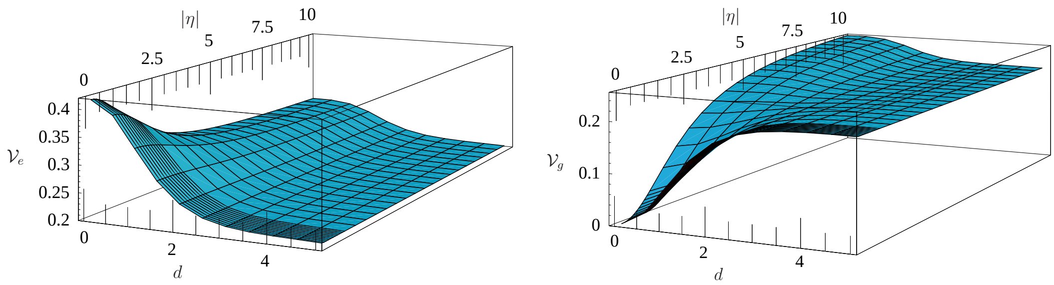


Figure 9: The Abelian potential $\mathcal{V}_e$ (left) and the non-Abelian potential $\mathcal{V}_g$ (right) for $\gamma = 1$.

5.1 Equal gauge coupling $\gamma = 1$ revisited

In section 4 we have discussed the effective potential for two vortices for any separation and with any relative orientation in a model with $\gamma = 1$ and $\lambda = \lambda_g = \lambda_e$. The potential is shown in figure 6. The effective potential comes from two pieces: the Abelian $\mathcal{V}_e$ and the non-Abelian $\mathcal{V}_g$ potential given in eq. (5.10). In figures 7 and 8 we show $\mathcal{V}_e$ and $\mathcal{V}_g$ taking several slices of figure 6.

Let us consider the case with $\lambda_e^2 - 1 > 0$ and $\lambda_g^2 - 1 > 0$. In this case the effective potential will have the same qualitative behaviors like the reduced potentials in the figures 7, 8 and 9. The figures shows how $\mathcal{V}_e$ and $\mathcal{V}_g$ behaves very differently. In particular,

	$\lambda_g^2 > 1$	$\lambda_g^2 = 1$	$\lambda_g^2 < 1$
$\lambda_e^2 > 1$	$N = (-,+)$ $A = +$	$N = 0$ $A = +$	$N = (+,-)$ $A = +$
$\lambda_e^2 = 1$	$N = (-,+)$ $A = 0$	$N = 0$ $A = 0$	$N = (+,-)$ $A = 0$
$\lambda_e^2 < 1$	$N = (-,+)$ $A = -$	$N = 0$ $A = -$	$N = (+,-)$ $A = -$

Table 3: The forces between two vortices in the $\gamma = 1$ case. N and A stand for non-Abelian force and Abelian force, respectively. 0 means no force, + means repulsive and $-$ means attractive. N=$(-,+)$ means that there is an attractive force for anti-parallel vortices and a repulsive force for parallel vortices.

the Abelian potential is always repulsive, both in the real and internal space[4] (see the red dots in figures 7 and 8). The non-Abelian potential is on the contrary sensitive on the orientations. In particular, figure 7 shows that it is repulsive for parallel vortices while it is attractive for anti-parallel ones. Furthermore, the non-Abelian potential becomes almost flat (along the spatial coordinate d) with the orientation $\eta \sim 4$, see the middle of figure 7. The blue dots in figure 8 reveal that the non-Abelian potential always gives attractive forces in the internal space. When the two scalar couplings are equal, $\lambda_e^2 = \lambda_g^2$, the left picture in figure 7 clearly shows how the two potentials exactly cancel for anti-parallel vortices, recovering the result of the previous section.

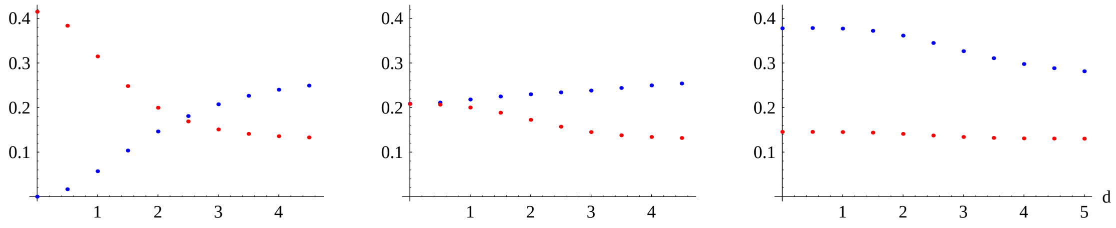
Of course, the true effective potential depends on λ_e and λ_g through the combination in eq. (5.11). This indicates the interaction between non-Abelian vortices is quite rich in comparison with that of the ANO vortices. Let us make a further example. The rightmost panel in figure 7 shows that $\mathcal{V}_e$ and $\mathcal{V}_g$ for parallel vortices ($\eta = \infty$) are identical[5]. Thus if we take $\lambda_e^2 + \lambda_g^2 = 2$, the interactions between parallel vortices vanishes while between anti-parallel vortices they do not canceled out. This is just opposite to what we found in the case $\lambda_e^2 = \lambda_g^2$. We summarize the possible behaviors of the Abelian and non-Abelian forces, when the gauge couplings are equal $e = g$ ($\gamma = 1$), in table 3.

5.2 Different gauge coupling $\gamma \neq 1$

We now consider interactions between non-Abelian vortices with different gauge coupling $e \neq g$ ($\gamma \neq 1$). Several numerical solutions are given in the left panel of figure 5. In figures 10 and 11 we show two numerical examples for the reduced effective potentials $\mathcal{V}_e$, $\mathcal{V}_g$ given in eq. (5.10).

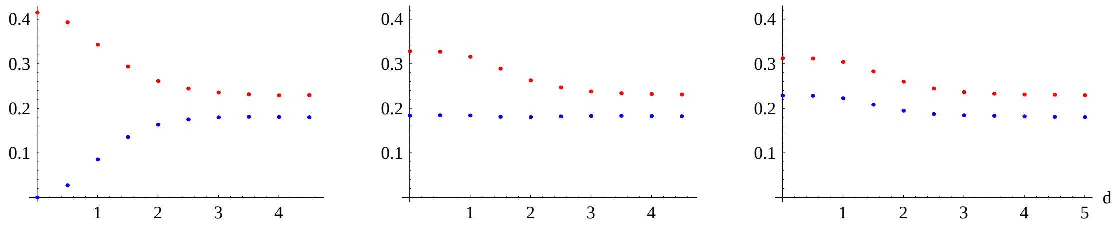
The plots show that the qualitative features of $\mathcal{V}_e$ and $\mathcal{V}_g$ are basically the same as what is discussed in the equal gauge coupling case ($\gamma = 1$). Therefore, the qualitative classification of the forces given in table 3 is still valid for $\gamma \neq 1$. We observe that the Abelian potential tends, at large distances, to a value smaller than the non-Abelian one for $\gamma < 1$, while opposite happens for $\gamma > 1$. The only things that have non dependence on

[4]Attraction in the internal space here just means that orientations tend to become the same.
[5]This statement can be proved analytically.



anti-parallel $(\eta = 0)$ intermediate $(\eta = 4)$ parallel $(\eta = \infty)$

Figure 10: Effective potential with $\gamma = 1/2$ vs. separation. (red, blue) $= (\mathcal{V}_e, \mathcal{V}_g)$.



anti-parallel $(\eta = 0)$ intermediate $(\eta = 4)$ parallel $(\eta = \infty)$

Figure 11: Effective potential with $\gamma = 1.3$ vs. separation. (red, blue) $= (\mathcal{V}_e, \mathcal{V}_g)$.

γ are the values of the potentials at (1,1)-vortices with $(d, \eta) = (0, 0)$. Regardless of the gauge couplings $\mathcal{V}_g(\eta = 0, d = 0) = 0$ while $\mathcal{V}_e(\eta = 0, d = 0) \fallingdotseq 0.41$. This is because the corresponding solution is proportional to the unit matrix, so that there are no contributions from the non-Abelian part.

The effective potential is obtained from the linear combination in eq. (5.11) and also depends on three parameters γ, λ_e and λ_g. With this big freedom we can obtain a lot of interesting interactions. For example, we can have potentials which develop a global minimum at some finite non zero distance. In such cases two vortices may be bounded at that distance. We can show a concrete potential in figure 12. The figure shows the presence of a minimum around $d \sim 2$.[6] This kind of behavior have not been found for the ANO type I/II vortices and the possibility of bounded vortices really results from the non-Abelian symmetry[7].

6. Interaction at large vortex separation

6.1 Vortices in fine-tuned models $e = g$ and $\lambda_e = \lambda_g$

In this subsection we will obtain an analytic formula for the asymptotic forces between vortices at large separation. We follow the technique developed in refs. [38, 43]. First of

[6]These kind of potentials are really generic. In fact, the presence of this minima does not require strong constraints on the couplings.

[7]Similar behaviors in the static intervortex potential were also observed for $\mathbb{Z}_3$ vortices in [37].

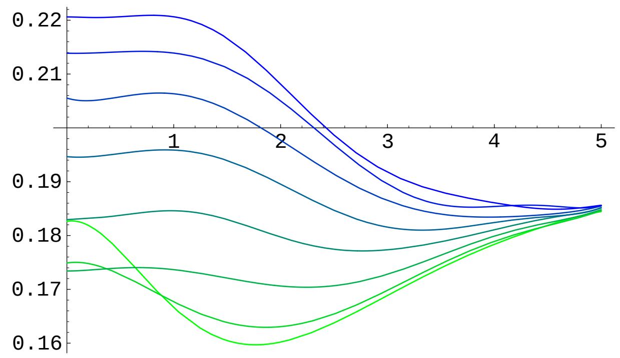


Figure 12: $\gamma = 1/2$, $\lambda_e = 1.2$, $\lambda_g = 1.06$: From $\eta = 0$ (green) to $\eta = 7$ (blue) with $d = 0 \sim 5$ for each η.

all, we need to find asymptotic behaviors of the scalar and the gauge fields. We again consider the $(1,0)$-vortex

$$H_0(z)^{(1,0)} = \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix}, \qquad \vec{\phi}_1^{(1,0)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{6.1}$$

When we are sufficiently far from the core of the vortex, $Y(r)$ and $f(r)$ in eqs. (4.2) and (4.3) can be written as

$$Y = 2\log r + \delta Y, \qquad f = 1 + \delta f \tag{6.2}$$

where δY and δf are small quantities. Plugging these into eqs. (4.2) and (4.3) and taking only linear terms in δY and δf, we obtain the following linearized equations

$$\delta Y'' + \frac{1}{r}\delta Y' - \lambda^2 \delta Y - 2\left(\delta f'' + \frac{1}{r}\delta f' - \lambda^2 \delta f\right) = 0, \tag{6.3}$$

$$\delta Y''' + \frac{1}{r}\delta Y'' - \frac{1}{r^2}\delta Y' - \delta Y' = 0. \tag{6.4}$$

Solutions to these equations are analytically obtained to be

$$\delta Y - 2\delta f = -\frac{q}{\pi}K_0(\lambda r), \quad \delta Y = \frac{m}{\pi}K_0(r) - C, \tag{6.5}$$

where $K_0(r)$ is the modified Bessel's function of zeroth order and q, m and C are integration constants. C must be 0 because $\delta Y \to 0$ as $r \to \infty$ while q, m should be determined by the original equation of motion. In the BPS case, $\delta f \equiv 0$ ($f \equiv 1$), thus $q = -m$. eq. (6.5) leads to the well known asymptotic behavior of the ANO vortex

$$H_{[1,1]} = f e^{-\frac{1}{2}Y} z \simeq \left(1 + \delta f - \frac{1}{2}\delta Y\right) e^{i\theta} = \left(1 + \frac{q}{2\pi}K_0(\lambda r)\right) e^{i\theta}, \tag{6.6}$$

$$\bar{W}_{[1,1]} = -\frac{i}{2}\bar{\partial} Y \simeq -\frac{i}{4}e^{i\theta}\frac{d}{dr}\left(2\log r + \delta Y\right) = -\frac{i}{2}\left(\frac{1}{r} - \frac{m}{2\pi}K_1(r)\right) e^{i\theta}, \tag{6.7}$$

where $K_1 \equiv -K_0'$ and we have defined $H_{[1,1]}$ and $\bar{W}_{[1,1]}$ as $[1,1]$ elements of H and $\bar{W}$ in eq. (3.7) with the $k=1$ ansatz (4.1).

Next we treat the vortices as point particles in a linear field theory coupled with a scalar source ρ and a vector current j_μ. To linearize the Yang-Mills-Higgs Lagrangian, we choose a gauge such that the Higgs fields is given by the following hermitian matrix

$$H = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} h^0+h^3 & h^1-ih^2 \\ h^1+ih^2 & h^0-h^3 \end{pmatrix}, \quad W_\mu = \frac{1}{2}\begin{pmatrix} w_\mu^0+w_\mu^3 & w_\mu^1-iw_\mu^2 \\ w_\mu^1+iw_\mu^2 & w_\mu^0-w_\mu^3 \end{pmatrix}. \tag{6.8}$$

Here all the fields h^a, w_μ^a are real. Then the quadratic part of the Lagrangian (3.2) is of the form

$$\mathcal{L}^{(2)}_{\text{free}} = \sum_{a=0}^{3}\left[-\frac{1}{4}f^a_{\mu\nu}f^{a\mu\nu} + \frac{1}{2}w^a_\mu w^{a\mu} + \frac{1}{2}\partial_\mu h^a \partial^\mu h^a - \frac{\lambda^2}{2}(h^a)^2\right] \tag{6.9}$$

where we have defined the Abelian field strength $f^a_{\mu\nu} \equiv \partial_\mu w^a_\nu - \partial_\nu w^a_\mu$. We also take into account the external source terms to realize the point vortex

$$\mathcal{L}_{\text{source}} = \sum_{a=0}^{3}\left[\rho^a h^a - j^a_\mu w^{a\mu}\right]. \tag{6.10}$$

The scalar and the vector sources should be determined so that the asymptotic behavior of the fields in eqs. (6.6) and (6.7) are replicated. Equations of motions are of the form

$$\left(\Box + \lambda^2\right) h^a = \rho^a, \qquad \left(\Box + 1\right) w^a_\mu = j^a_\mu. \tag{6.11}$$

In order to replicate the $(1,0)$-vortex corresponding to the $k=1$ ansatz (4.1), we just need to mimic the result of refs. [38, 43] because the single non-Abelian vortex is a mere embedding of the ANO vortex as mentioned earlier. In fact, only $(h^0, w^0_\mu, \rho^0, j^0_\mu) = (h^3, w^3_\mu, \rho^3, j^3_\mu)$ are relevant and all the others are zero:

$$\begin{aligned} h^0 = h^3 &= \frac{q}{2\pi}K_0(\lambda r), & \rho^0 = \rho^3 &= q\delta(r), \\ \mathbf{w}^0 = \mathbf{w}^3 &= -\frac{m}{2\pi}\hat{\mathbf{k}} \times \nabla K_0(r), & \mathbf{j}^0 = \mathbf{j}^3 &= -m\hat{\mathbf{k}} \times \nabla\delta(r) \end{aligned} \tag{6.12}$$

where $\hat{\mathbf{k}}$ is a spatial fictitious unit vector along the vortex world-volume. The vortex configuration with general orientation is also treated easily, since the origin of the orientation is the Nambu-Goldstone mode associated with the broken SU(2) color-flavor symmetry

$$H_0 \to H_0(z)^{(1,0)}U_{\rm F}, \quad \vec{\phi}_2 = U_{\rm F}^\dagger \vec{\phi}_1^{(1,0)} = \begin{pmatrix} \alpha^* \\ \beta^* \end{pmatrix}, \quad U_{\rm F} \equiv \begin{pmatrix} \alpha & \beta \\ -\beta^* & \alpha^* \end{pmatrix}, \tag{6.13}$$

where $|\alpha|^2 + |\beta|^2 = 1$. The fields H and W_μ receive the following transformations, keeping the hermitian form of (6.8).

$$\begin{pmatrix} X & 0 \\ 0 & 0 \end{pmatrix} \to U_{\rm F}^\dagger \begin{pmatrix} X & 0 \\ 0 & 0 \end{pmatrix} U_{\rm F} = \begin{pmatrix} |\alpha|^2 & \alpha^*\beta \\ \alpha\beta^* & |\beta|^2 \end{pmatrix} X. \tag{6.14}$$

The scalar interaction between a vortex at $\mathbf{x} = \mathbf{x}_1$ with the orientation $\vec{\phi}_1$ and another vortex at $\mathbf{x} = \mathbf{x}_2$ with the orientation $\vec{\phi}_2$ can be obtained by

$$\begin{aligned} L_h &= \int dx^2 \ \mathrm{Tr}\left[\begin{pmatrix} h^0(\mathbf{x}-\mathbf{x}_1) & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} |\alpha|^2 & \alpha^*\beta \\ \alpha\beta^* & |\beta|^2 \end{pmatrix}\rho^0(\mathbf{x}-\mathbf{x}_2)\right] \\ &= |\alpha|^2 \frac{q^2}{2\pi} K_0(\lambda|\mathbf{x}_1 - \mathbf{x}_2|). \end{aligned} \tag{6.15}$$

The gauge interaction is also obtained by similar way

$$\begin{aligned} L_w &= -\int dx^2 \ \mathrm{Tr}\left[\begin{pmatrix} \mathbf{w}^0(\mathbf{x}-\mathbf{x}_1) & 0 \\ 0 & 0 \end{pmatrix}\cdot\begin{pmatrix} |\alpha|^2 & \alpha^*\beta \\ \alpha\beta^* & |\beta|^2 \end{pmatrix}\mathbf{j}^0(\mathbf{x}-\mathbf{x}_2)\right] \\ &= -|\alpha|^2 \frac{m^2}{2\pi} K_0(|\mathbf{x}_1 - \mathbf{x}_2|). \end{aligned} \tag{6.16}$$

Then total potential is $V_{\rm int} = -L_h - L_w$

$$V_{\rm int} = -\frac{\left|\vec{\phi}_1^\dagger \vec{\phi}_2\right|^2}{\left|\vec{\phi}_1\right|^2\left|\vec{\phi}_2\right|^2}\left(\frac{q^2}{2\pi}K_0(\lambda|\mathbf{x}_1-\mathbf{x}_2|) - \frac{m^2}{2\pi}K_0(|\mathbf{x}_1-\mathbf{x}_2|)\right), \tag{6.17}$$

where $|\alpha|^2 = \frac{\left|\vec{\phi}_1^\dagger \vec{\phi}_2\right|^2}{\left|\vec{\phi}_1\right|^2\left|\vec{\phi}_2\right|^2}$ is invariant under the global color-flavor rotation. When two vortices have parallel orientations, this potential becomes that of two ANO vortices [38]. On the other hand, the potential vanishes when their orientations are anti-parallel. This agrees with the numerical result found in the previous sections. In the BPS limit $\lambda = 1$ $(q = m)$, the interaction becomes precisely zero.

Since $K_0(\lambda r) \sim \sqrt{\pi/2\lambda r}e^{-\lambda r}$, the potential asymptotically reduces to

$$V_{\rm int} \simeq \begin{cases} -\frac{\left|\vec{\phi}_1^\dagger \vec{\phi}_2\right|^2}{\left|\vec{\phi}_1\right|^2\left|\vec{\phi}_2\right|^2}\frac{q^2}{2\pi}\sqrt{\frac{\pi}{2\lambda r}}e^{-\lambda r} & \text{for } \lambda < 1 \ \text{Type I} \\ \frac{\left|\vec{\phi}_1^\dagger \vec{\phi}_2\right|^2}{\left|\vec{\phi}_1\right|^2\left|\vec{\phi}_2\right|^2}\frac{m^2}{2\pi}\sqrt{\frac{\pi}{2r}}e^{-r} & \text{for } \lambda > 1 \ \text{Type II} \end{cases} \tag{6.18}$$

where $r \equiv |\mathbf{x}_1 - \mathbf{x}_2| \gg 1$. If we fix the relative orientation being some finite value, the force $F_r = -\partial_r V_{\rm int}$ between two vortices is attractive for $\lambda < 1$ and repulsive for $\lambda > 1$ similar to the force between ANO vortices. The force vanishes when the relative orientation becomes anti-parallel. If we fix the distance by hand, the orientations tend to be anti-parallel for the type II while the parallel configuration are preferred for the type I case.

6.2 Vortices with general couplings

It is quite straightforward to generalize the results of the previous section to the case of generic couplings. We can of course use the same gauge as in eq. (6.8). The quadratic Lagrangian (2.6) is of the form

$$\begin{aligned} \mathcal{L}^{(2)}_{\rm free} &= \sum_{a=1}^{3}\left[-\frac{1}{4\gamma^2}f^a_{\mu\nu}f^{a\mu\nu} + \frac{1}{2}w^a_\mu w^{a\mu} + \frac{1}{2}\partial_\mu h^a \partial^\mu h^a - \frac{\lambda_g^2\gamma^2}{2}(h^a)^2\right] \\ &\quad + \left[-\frac{1}{4}f^0_{\mu\nu}f^{0\mu\nu} + \frac{1}{2}w^0_\mu w^{0\mu} + \frac{1}{2}\partial_\mu h^0 \partial^\mu h^0 - \frac{\lambda_e^2}{2}(h^0)^2\right]. \end{aligned} \tag{6.19}$$

The external sources can be still reproduced by source terms as in eqs. (6.10), (6.11). The linearized equations following from the above Lagrangian are of the form

$$\left(\frac{1}{\gamma^2}\Box+1\right)w^a_\mu=j^a_\mu,\quad \left(\Box+\lambda_g^2\gamma^2\right)h^a=\rho^a,\quad \left(\Box+1\right)w^0_\mu=j^0_\mu,\quad \left(\Box+\lambda_e^2\right)h^0=\rho^0 \tag{6.20}$$

For the $(1,0)$-vortex the only non-zero profile functions are $(h^0,w^0_\mu,\rho^0,j^0_\mu)$ and $(h^3,w^3_\mu,\rho^3,j^3_\mu)$. But these profiles are no longer equal and we need to deal with them independently. The only difference from the similar equations (6.11) is for the masses of the particles. The masses are directly related to asymptotic tails of vector and scalar fields. We can easily find the solutions by doubling eqs. (6.12)

$$\begin{aligned}
h^0&=\frac{q^0}{2\pi}K_0(\lambda_e r), & \mathbf{w}^0&=-\frac{m^0}{2\pi}\hat{\mathbf{k}}\times\nabla K_0(r),\\
\rho^0&=q^0\,\delta(r), & \mathbf{j}^0&=-m^0\,\hat{\mathbf{k}}\times\nabla\delta(r), \qquad (6.21)\\
h^3&=\frac{q^3}{2\pi}K_0(\lambda_g\gamma r), & \mathbf{w}^3&=-\frac{m^3}{2\pi}\hat{\mathbf{k}}\times\nabla K_0(\gamma r),\\
\rho^3&=q^3\,\delta(r), & \mathbf{j}^3&=-m^3\,\hat{\mathbf{k}}\times\nabla\delta(r). \qquad (6.22)
\end{aligned}$$

The vortex with the orientation $\vec{\phi}_2$ in eq. (6.13) can be obtained by performing an $\mathrm{SU}(2)_{\mathrm{C+F}}$ rotation like eq. (6.14)

$$\begin{aligned}
&\begin{pmatrix}\frac{X^0+X^3}{2} & 0\\ 0 & \frac{X^0-X^3}{2}\end{pmatrix} \qquad (6.23)\\
&\to\begin{pmatrix}\frac{X^0}{2}+\left(|\alpha|^2-|\beta|^2\right)\frac{X^3}{2} & \left(\alpha^*\beta-\beta\alpha^*\right)\frac{X^0}{2}+\left(\alpha^*\beta+\beta\alpha^*\right)\frac{X^3}{2}\\ \left(\alpha\beta^*-\beta^*\alpha\right)\frac{X^0}{2}+\left(\alpha^*\beta+\beta\alpha^*\right)\frac{X^3}{2} & \frac{X^0}{2}-\left(|\alpha|^2-|\beta|^2\right)\frac{X^3}{2}\end{pmatrix}.
\end{aligned}$$

Similar to eqs. (6.15) and (6.16), we find the total potential V_{int}

$$\begin{aligned}
V_{\mathrm{int}}&=\frac{1}{2}\left(-\frac{(q^0)^2}{2\pi}K_0(\lambda_e|\mathbf{x}_1-\mathbf{x}_2|)+\frac{(m^0)^2}{2\pi}K_0(|\mathbf{x}_1-\mathbf{x}_2|)\right) \qquad (6.24)\\
&\quad+\left(\frac{\left|\vec{\phi}_1^\dagger\vec{\phi}_2\right|^2}{\left|\vec{\phi}_1\right|^2\left|\vec{\phi}_2\right|^2}-\frac{1}{2}\right)\left(-\frac{(q^3)^2}{2\pi}K_0(\lambda_g\gamma|\mathbf{x}_1-\mathbf{x}_2|)+\frac{(m^3)^2}{2\pi}K_0(\gamma|\mathbf{x}_1-\mathbf{x}_2|)\right).
\end{aligned}$$

When we tune the parameters to $\gamma=1$ and $\lambda_g=\lambda_e$ ($q^0=q^3,\ m^0=m^3$), this effective potential is exactly identical to that of eq. (6.17). In the BPS limit $\lambda_e=\lambda_g=1$, the interaction becomes precisely zero because $q^0=m^0$ and $q^3=m^3$.

At large distance, the interactions between vortices are dominated by the particles with the lowest mass $M_{\rm low}$. There are four possible regimes

$$V_{\rm int} = \begin{cases} -\dfrac{(q^0)^2}{4\pi}\sqrt{\dfrac{\pi}{2\lambda_e r}}e^{-\lambda_e r} & \text{for } M_{\rm low} = M_{\rm s}, \quad \text{Type I} \\ -\left(\dfrac{\left|\vec{\phi}_1^\dagger \vec{\phi}_2\right|^2}{\left|\vec{\phi}_1\right|^2\left|\vec{\phi}_2\right|^2} - \frac{1}{2}\right)\dfrac{(q^3)^2}{2\pi}\sqrt{\dfrac{\pi}{2\lambda_g \gamma r}}e^{-\lambda_g\gamma r} & \text{for } M_{\rm low} = M_{\rm ad}, \quad \text{Type I}^* \\ \dfrac{(m^0)^2}{4\pi}\sqrt{\dfrac{\pi}{2r}}e^{-r} & \text{for } M_{\rm low} = M_{\rm U(1)}, \quad \text{Type II} \\ \left(\dfrac{\left|\vec{\phi}_1^\dagger \vec{\phi}_2\right|^2}{\left|\vec{\phi}_1\right|^2\left|\vec{\phi}_2\right|^2} - \frac{1}{2}\right)\dfrac{(m^3)^2}{2\pi}\sqrt{\dfrac{\pi}{2\gamma r}}e^{-\gamma r} & \text{for } M_{\rm low} = M_{\rm SU(2)}, \ \text{Type II}^* \end{cases} \tag{6.25}$$

which exhaust all the possible kinds of asymptotic potentials of this system. This generalizes the type I/II classification of Abelian superconductors. We have found two new categories, called type I* and type II*, in which the force can be attractive or repulsive depending on the relative orientation. The type I force is always attractive and the type II force is repulsive regardless of the relative orientation. On the other hand, type I* and type II* depend on the relative orientation. In the type I* case the forces between parallel vortices are attractive while anti-parallel vortices repel each other. The type II* vortices feel opposite forces to the type I*[8]. Note that we have used the same terms type I/II for the fine-tuned model in section 6.1. In the perspective of this section, they should be called type I+I*/II+II* because of the degeneracy of some masses. It is interesting to compare these results with the recently studied asymptotic interactions between non-BPS non-Abelian global vortices [23]. A very different feature of global vortices is that the interactions are always repulsive. This is because they are mediated by Nambu-Goldstone zero modes, whereas in our model these particles are all eaten by the gauge bosons thanks to the Higgs mechanism.

We find a nice matching of qualitative features between the numerical results of the previous sections and the semi-analytical results in this section. Let us look at figures 7, 10 and 11. In all the cases, we found that the Abelian potentials are attractive regardless of the orientations while the non-Abelian potentials are sensitive to those. These properties are well shown also in the semi-analytical results in (6.25). The type I/II interactions originated by the U(1) part are independent of the orientations whereas the type I*/II* which are coming from the SU(N) part do depend on them.

The result in eq. (6.25) is easily extended to the general case of U(1) $\times$ SU(N). This can be done by just thinking of the orientation vectors $\vec{\phi}$ as taking values in $\mathbf{C}P^{N-1}$.

7. Conclusion and discussion

In this paper we have studied static interactions between non-BPS vortices in SU(N)$\times$U(1) gauge theories with Higgs fields in the fundamental representation. We have discussed

[8] In ref. [41] we have found a similar result in a supersymmetric theory. The relation between the two notations is explained in the appendix. The supersymmetric theory in ref. [41] shows only type I/I* behaviors.

models with arbitrary gauge and scalar couplings. We have numerically computed the effective potential for almost BPS configurations for arbitrary separations and any internal orientations. We have also obtained analytic expressions for the static forces between well separated non-BPS vortices. This expression is valid also for models far from BPS limit.

For the fine-tuned model we found interaction pattern similar to that of the ANO vortices in the Abelian-Higgs model. The numerical effective potential is given in figure 6, it depends on both the relative distance and the orientations. The asymptotic potential between two vortices is given by eq. (6.18). In this model the mass of the U(1) and of the SU(N) vector bosons are same $M_{\mathrm{SU}(N)} = M_{\mathrm{U}(1)}$, and also all the scalars have the same masses $M_{\mathrm{s}} = M_{\mathrm{ad}}$. We thus have only two mass scales, which corresponds to two different asymptotic regimes. For $\lambda < 1$ ($M_{\mathrm{s}} < M_{\mathrm{U}(1)}$) there is universal attraction (type I) and for $\lambda > 1$ ($M_{\mathrm{s}} > M_{\mathrm{U}(1)}$) universal repulsion (type II). Both the numerical and the analytical result show that the interactions between two anti-parallel vortices vanish; this configuration is unstable for type I vortices and stable for type II. So in this last case the part of the moduli space which corresponds to vortices with opposite $\mathbf{C}P^1$ orientations at arbitrary distance survives the non-BPS perturbation.

In models with arbitrary couplings, on the other hand, the pattern of interactions becomes richer. In this case we considered separately the Abelian and non-Abelian contributions $\mathcal{V}_e$ and $\mathcal{V}_g$ to the effective potential. The two show very different qualitative behavior. While the Abelian contribution is always attractive (or repulsive) for a given choice of the parameters, the non-Abelian one can be attractive or repulsive depending on the relative internal orientation of the two vortices. These properties combined with the fact that the full effective potential is the linear combination given in eq. (5.11) deduce that we can obtain many qualitatively different type of interactions depending on the choice of the parameters of the theory. Such a variety also appears in the possible asymptotic behavior of the interactions. In the theory there are four different mass scales, the masses of the SU(N) vector bosons $M_{\mathrm{SU}(N)}$, the U(1) vector boson $M_{\mathrm{U}(1)}$, the adjoint scalars M_{ad} and the singlet scalar M_{s} under the color-flavor symmetry. This leads to four different asymptotic regimes, classified in eq. (6.25). We found, in addition to type I and type II, new types of interaction mediated by the non-Abelian particles, which we call type I* and type II*. The type I (type II) force is attractive (repulsive), and occurs when the singlet scalar (U(1) vector boson) has the smallest mass. These forces do not depend on the relative orientation. In the type I* case there is an attractive force for parallel orientations and repulsive for anti-parallel ones where the asymptotic force is mediated by the lightest non-Abelian scalar fields. On the other hand for type II* there is repulsion for parallel orientation and attraction for anti-parallel ones and the force at large distance is mediated by the lightest non-Abelian vector field.

The dynamics of the interactions of the non-BPS non-Abelian vortices is quite rich. Let us give comments on some possible further directions:

- *Reconnection rate of the cosmic string:* The slow moving non-Abelian BPS vortex strings, as cosmic strings, were shown to always reconnect with probability one [17, 18]. This is important in order to distinguish solitonic cosmic strings from fundamental cosmic strings which generically have very small reconnection probability. Consider the two dimensional space spanned by η and d; the coordinate d corresponds to the vortices relative distance and η to the internal colour-flavour orientation. The $(2,0)$-vortices are at $\eta \to \infty$ and the $(1,1)$-vortices at $\eta = 0$. Only the orbits which pass through the point $d = 0, \eta = 0$ correspond to scatterings where no reconnection occurs; these orbits represent scatterings with very finely tuned initial conditions. The fine-tuned scattering process cannot contribute to the reconnection probability; for this reason the reconnection rate of the non-Abelian BPS vortex is one [17, 18].

When we consider strong non-BPS corrections, the moduli space approximation is no longer valid and this conclusion could drastically change. For some values of the couplings there will appear regimes in which the coincident $(1,1)$-vortices are favored to the coincident $(2,0)$-vortices. In that case we may expect that the reconnection probability becomes smaller than one. As a very simple example, we can consider the fine tuned model for $\lambda > 1$. In this case the energetically favored configuration is the one with two vortices with opposite $\mathbf{C}P^1$ orientation, which never reconnect. In this case we can expect a reduction of the reconnection rate. On the contrary for $\lambda < 1$, when the $(2,0)$ state is energetically favored, we expect that the reconnection rate should still be one. It would be interesting to make a detailed numerical investigation of the scattering process of non-BPS non-Abelian vortices, in order to clarify how the non-BPS corrections could modify the reconnection rate.

- *Non-Abelian vortices and Abrikosov lattice:* In the usual type II superconductor (the Abelian-Higgs model), if a large number of vortices penetrate a region of given area A, they will form a hexagonal lattice (Abrikosov lattice) rather than forming a square one [39]. This is verified experimentally. If we consider the same for non-Abelian vortices, we expect that the property of the lattice can be quite different, say, lattice spacing and/or form can change. An interesting possibility is the appearance of phase transitions due to the change of the lattice structure when the density of vortices change. This eventuality is suggested by the possible presence of interactions which change with distances. For example in the case of figure 12 there are repulsive forces at short distances while at large distances they are attractive.

- *Quantum aspects:* In this paper we focused on the classical aspects of the interactions between non-Abelian vortices. In the theoretical set-up that we have discussed, the quantum aspects of the infrared physics of a single vortex are described by an effective bosonic $\mathbf{C}P^{N-1}$ sigma model (the theoretical setting is in this sense similar to the one discussed ref. [7]). Let us consider two vortices at large distance; in the type I* and in the type II* regimes the quantum physics will be described by two $\mathbf{C}P^{N-1}$ sigma models with an interaction potential given by eq. (6.25). It would be interesting to study the effect of this term in the sigma model physics. Another interesting problem is the numerical determination of the effective theory for vortices at generic separations. To do this one has to determine the Manton metric on the full moduli space.

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A. Note on the relation between two formalisms

The aim of this small section is to relate the moduli matrix formalism that we used in this paper to the direct ansatz approach we choose in a previous work [41]. To this end we must find the relation between the angle α used in [41] and the moduli matrix parameters $\eta = 1/a'$. The resulting relation between the two variables, in the case of coincident vortices, is rather non trivial.

Coincident vortices. In the moduli matrix formalism, two coincident vortices are described by the following moduli matrix

$$H_0^{(1,1)} = \begin{pmatrix} z & -\eta \\ 0 & z \end{pmatrix}, \tag{A.1}$$

while in [41] we used explicit ansatzs for the fields. In particular, for the squarks field we have used

$$H_{\text{ans}} = \begin{pmatrix} -\cos\frac{\alpha}{2} e^{2i\varphi}\kappa_1(r) & \sin\frac{\alpha}{2} e^{i\varphi}\kappa_2(r) \\ -\sin\frac{\alpha}{2} e^{i\varphi}\kappa_3(r) & -\cos\frac{\alpha}{2}\kappa_4(r) \end{pmatrix}. \tag{A.2}$$

This form for the squark fields lead us, in [11, 12], to conjecture that this solution is associated with that given by the following moduli matrix:

$$H_{\text{ans}} \quad \Leftrightarrow \quad H_{0,\text{ans}} = \begin{pmatrix} -\cos\frac{\alpha}{2} z^2 & \sin\frac{\alpha}{2} z \\ -\sin\frac{\alpha}{2} z & -\cos\frac{\alpha}{2} \end{pmatrix}. \tag{A.3}$$

Using a V equivalence we can put $H_{0,ans}$ on the standard $(1,1)$ form

$$H_0^{(1,1)} = V(z) H_{0,\text{ans}} = \begin{pmatrix} z & \cot(\alpha/2) \\ 0 & z \end{pmatrix}. \tag{A.4}$$

This simple argument gave us the relation: $\eta = \cot(\alpha/2)$.

Here we point out that this conjecture is wrong. In fact the functions $(\kappa_1, \kappa_2, \kappa_3, \kappa_4)$ in eq. (A.2) have an implicit dependence on α, which come out only after solving the differential equations for the vortices. This means that it is difficult to find a simple analytical relation between the parameters of the two formalisms. Also, this relation is not the same for all the values of $\gamma = g/e$. In figure 13 we plotted the numerical relation between α and η for $\gamma = 1$. The corresponding values for the two variables are found making an

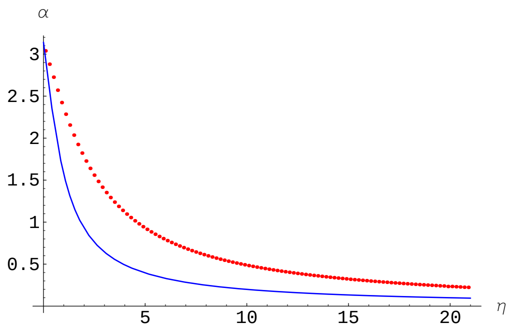

Figure 13: The solid line is the naive relation $\eta = \cot(\alpha/2)$ and the dots are the true numerical relations between η and α for $\gamma = 1$

empirical match of some gauge invariant functions which have a non-trivial dependence on the relative orientation. One of these functions, $\mathrm{Tr}F_{12}$, is showed in figure 3. The fact that a perfect matching is possible between gauge invariant functions in both the formalisms is a strong numerical check of the consistence of the two approaches.

Well separated vortices. Vortices at large separation have well-defined global orientations in the internal space, and thus a well-defined notion of relative orientation. This enables us to find an exact relation between the parameters of the two formalisms. In [41] the global orientation of a non-Abelian vortex was defined by a vector which takes values in the internal space: $\vec{n}$. Using global color-flavor transformations we can always put the orientations $\vec{n}_1$ and $\vec{n}_2$ of two vortices on the following standard form

$$\vec{n}_1 = (0,0,1), \quad \vec{n}_2 = (-\sin\alpha, 0, \cos\alpha). \tag{A.5}$$

The vector $\vec{n}_2$ can be obtained from $\vec{n}_1$ acting with a global rotation

$$\vec{n}_2 \cdot \vec{\tau} \equiv U^{-1}\tau^3 U, \quad \text{with} \quad U = \begin{pmatrix} \cos\frac{\alpha}{2} & -\sin\frac{\alpha}{2} \\ \sin\frac{\alpha}{2} & \cos\frac{\alpha}{2} \end{pmatrix}. \tag{A.6}$$

We can repeat the same argument for the orientation vectors defined within the moduli matrix formalism (see eq. (3.18)). We can choose the following identification:

$$(\vec{n}_1, \vec{n}_2) = ((0,0,1), (-\sin\alpha, 0, \cos\alpha)) \quad \Leftrightarrow \quad (\vec{\phi}_1, \vec{\phi}_2) = \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ b' \end{pmatrix} \right). \tag{A.7}$$

If we act with the same global rotation on $\vec{\phi}_1$

$$\vec{\phi}_2 = U^\dagger \vec{\phi}_1 = \begin{pmatrix} \cos\frac{\alpha}{2} \\ -\sin\frac{\alpha}{2} \end{pmatrix} \sim \begin{pmatrix} 1 \\ -\tan\frac{\alpha}{2} \end{pmatrix}, \tag{A.8}$$

we find the relation $b' = -\tan\frac{\alpha}{2}$. With this identification we can easily check the identity:

$$\frac{\left|\vec{\phi}_1^\dagger \vec{\phi}_2\right|^2}{\left|\vec{\phi}_1\right|^2 \left|\vec{\phi}_2\right|^2} - \frac{1}{2} = \frac{\vec{n}_1 \cdot \vec{n}_2}{2} = \frac{\cos\alpha}{2}. \tag{A.9}$$

This shows the consistence of the expressions for the asymptotic forces obtained in this paper in eq. (6.25) with those found in [41].

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